On Toric Varieties

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Preface

I first learned about toric varieties when I read Stanley's proof of McMullen's famous g-conjecture (see [10]). In this proof Stanley interprets the components of the h-vector of a rational simplicial convex polytope as the dimensions of the cohomology groups of the associated toric variety. Then, by applying the hard Lefschetz theorem to the toric variety and using results from commutative algebra, he deduces the conjectured conditions for the h-vector. It is this connection between the topology of toric varieties and the combinatorial geometry of convex polytopes (or fans in general) which has fascinated me since then.

The idea of this thesis was to describe the cohomology (with integral coefficients) of a toric variety in terms of the underlying fan which already contains all of the information. To this aim we establish in section 2 a spectral sequence for toric varieties whose E_2-term is determined, up to multilinear algebra, by the cones of the underlying fans. As an application we show how this spectral sequence can be used to compute the cohomology of a 3-dimensional toric variety. Before, in section 1, we present a topological definition of toric varieties due to R. MacPherson and construct a finite CW-cell decomposition. Finally, in section 3, we give a topological classification of the smooth 2-dimensional toric varieties. This section, which is joint work with David Yavin, may also serve as a source of examples of toric varieties.

At this point I wish to thank all the people who have supported me during the time when I was writing this thesis. Especially I am indebted to Peter Mani, my advisor, for having aroused my interest in many mathematical problems (as well as in non-mathematical things like Charlie Parker). It is also due to him that I met some other interesting mathematicians like Prof. Ewald and Markus Eikelberg who introduced to us the algebraic geometry of toric varieties, and David Yavin who gave us an insight into their topological structure. The resulting collaboration with David let me experience how lively mathematics can be. I also wish to thank Victor Batyrev and Urs Würgler for the stimulating discussions I had with them, and the National Science Foundation for the financial support which made this thesis possible.

List of symbols

B^d	unit ball in $\mathbf{R}^d$
S^{d-1}	unit sphere in $\mathbf{R}^d$
Σ	complete fan in $\mathbf{R}^d$
σ	cone of the fan Σ
$\hat{\sigma}$	cell of B^d dual to the cone σ
$\sigma^\perp$	subspace of $\mathbf{R}^d$ orthogonal to the cone σ
$\sigma^\vee$	subgroup of $\mathbf{Z}^d$ orthogonal to the cone σ
T^d	d-dimensional (real) torus
T_σ	subtorus of T^d generated by the cone σ
$\hat{T}_\sigma$	quotient of T^d modulo the subtorus T_σ
$p_{\tau,\sigma}$	canonical projection of $\hat{T}_\tau$ onto $\hat{T}_\sigma$ (for $\tau \subset \sigma$)
X_Σ	toric variety associated to the fan Σ
p	projection of X_Σ onto B^d

By the interior and the boundary of a subspace $A \subset X$, denoted by $\operatorname{int} A$ and ∂A, we always mean its relative interior, respectively boundary, with respect to its closure $\operatorname{cl} A$ in X.

Contents

1 Preliminaries

1.1 Definitions

A set $\sigma \subset \mathbf{R}^d$ is called a rational polyhedral cone in $\mathbf{R}^d$, if there exist finitely many vectors $v_1, \ldots, v_n \in \mathbf{Z}^d$ such that $\sigma = \mathbf{R}_{\geq 0}v_1 + \ldots + \mathbf{R}_{\geq 0}v_n$ (where $\mathbf{R}_{\geq 0}$ denotes the non-negative real numbers). The dimension of σ is defined to be the dimension of the subspace $\operatorname{span}\sigma$, and σ is called simplicial if the generating vectors $v_1, \ldots, v_n$ can be chosen linearly independent (and hence $n = \dim \sigma$). If $H \subset \mathbf{R}^d$ is a hyperplane which contains the origin $0 \in \mathbf{R}^d$ such that σ lies in one of the closed halfspaces of $\mathbf{R}^d$ bounded by H, then the intersection $\sigma \cap H$ is again a rational polyhedral cone which is called a face of σ. If $\{0\}$ is a face of σ, we say that σ has a vertex at 0.

Definition 1.1 A *fan* Σ in $\mathbf{R}^d$ is a finite set of rational polyhedral cones in $\mathbf{R}^d$ satisfying the following conditions:
(i) Every cone $\sigma \in \Sigma$ has a vertex at 0;
(ii) If τ is a face of a cone $\sigma \in \Sigma$, then $\tau \in \Sigma$;
(iii) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face both of σ and σ'.
Σ is called simplicial if it consists of simplicial cones, and Σ is called complete if $\bigcup_{\sigma\in\Sigma} \sigma = \mathbf{R}^d$. Let $\Sigma^{(i)}$ denote the set of i-dimensional cones in Σ and a_i their number.

Example 1.1 Let $P \subset \mathbf{R}^d$ be a d-dimensional convex polytope with rational vertices and assume that $0 \in \operatorname{int} P$. Then the cones $\bigcup_{x\in F} \mathbf{R}_{\geq 0}x$ spanned by the faces F of P, together with the cone $\{0\}$, form a complete fan in $\mathbf{R}^d$ which is simplicial whenever the polytope P is simplicial. Such a fan which is generated by a convex polytope is called polytopal.

In the following we will always assume a fan to be complete. Thus the sets obtained by intersecting each cone $\sigma \in \Sigma$, $\sigma \neq \{0\}$, with the unit sphere $S^{d-1} \subset \mathbf{R}^d$ form a spherical complex $\mathcal{C}$. Let $\mathcal{C}'$ be the barycentric subdivision of $\mathcal{C}$, and for $\sigma \in \Sigma$, $\sigma \neq \{0\}$, let $\hat{\sigma}$ be the union of all simplices in $\mathcal{C}'$ whose vertices are barycenters of elements in $\mathcal{C}$ which contain $\sigma \cap S^{d-1}$. Note that $\operatorname{int}\hat{\sigma}$ is a homology cell of dimension $d - \dim \sigma$. For $\sigma = \{0\}$ we set $\hat{\sigma} = B^d$, the unit ball of $\mathbf{R}^d$, and call $\hat{\Sigma} = \{\hat{\sigma} | \ \sigma \in \Sigma\}$ the dual complex of Σ.

Let T^d denote the d-dimensional torus $\mathbf{R}^d/\mathbf{Z}^d$. Each k-dimensional cone $\sigma \in \Sigma$ spans a k-dimensional subspace of $\mathbf{R}^d$ which, since it has a rational basis, maps under the canonical projection $\mathbf{R}^d \to T^d$ to the k-dimensional subtorus

$$T_\sigma = (\operatorname{span}\sigma + \mathbf{Z}^d)/\mathbf{Z}^d \cong \operatorname{span}\sigma/(\operatorname{span}\sigma \cap \mathbf{Z}^d).$$

Definition 1.2 The *toric variety* X_Σ associated to the complete fan Σ in $\mathbf{R}^d$ is defined to be the quotient space $B^d \times T^d/\sim$ where $\sim$ is the equivalence relation given by

$$(x,t) \sim (x',t') \iff x = x' \text{ and } t - t' \in T_\sigma \text{ where } x \in \operatorname{int}\hat{\sigma}.$$

We call d the dimension of X_Σ and denote by p the projection of X_Σ onto B^d.

Note that over each interior point of a face $\hat{\sigma} \subset B^d$ the torus T^d is collapsed by the relation $\sim$ to the quotient

$$\hat{T}_\sigma = T^d/T_\sigma \cong \mathbf{R}^d/(\operatorname{span}\sigma + \mathbf{Z}^d).$$

Since $\operatorname{span}\sigma \cap \mathbf{Z}^d$ is a direct summand of $\mathbf{Z}^d$, i.e. there is a subgroup $G \subset \mathbf{Z}^d$ such that $\mathbf{Z}^d = (\operatorname{span}\sigma \cap \mathbf{Z}^d) \oplus G$, it follows that $\hat{T}_\sigma \cong (\operatorname{span} G)/G$ is a torus of dimension $d - \dim\sigma = \dim\hat{\sigma}$ and $T^d \cong T_\sigma \oplus \hat{T}_\sigma$.

The proofs of the following properties can be found e.g. in [1].

Theorem 1.1 *Let X_Σ be the toric variety associated to the complete fan Σ in $\mathbf{R}^d$.*
(i) X_Σ is compact and simply connected.
(ii) X_Σ is a smooth manifold if and only if Σ is simplicial and every d-dimensional cone in Σ can be generated by a basis of $\mathbf{Z}^d$ (in which case the fan Σ is called regular).
(iii) If f is a unimodular transformation of $\mathbf{R}^d$, then the toric variety $X_{\Sigma'}$ associated to the fan $\Sigma' = \{f(\sigma) | \sigma \in \Sigma\}$ is homeomorphic to X_Σ.

1.2 CW-cell decomposition

Next we construct a finite CW-cell decomposition of a toric variety X_Σ. For each pair of incident cones $\tau \subset \sigma$ in the underlying fan Σ let $p_{\tau,\sigma}$ denote the canonical projection $\hat{T}_\tau \to \hat{T}_\sigma$. Then X_Σ is homeomorphic to the topological sum $\sum_{\sigma\in\Sigma} \hat{\sigma} \times \hat{T}_\sigma$ in which two points are identified whenever they correspond under a map $\mathrm{id}_{B^2} \times p_{\tau,\sigma}$. Therefore, if we define for each cone $\sigma \in \Sigma$ a CW-cell decomposition of the torus $\hat{T}_\sigma$ such that the projection maps $p_{\tau,\sigma}$ are all cellular, the products of the cells $\operatorname{int}\hat{\sigma}$ with the cells in the decomposition of the corresponding torus $\hat{T}_\sigma$ form a CW-cell decomposition of the toric variety

$$X_\Sigma = \bigcup_{\sigma\in\Sigma} \operatorname{int}\hat{\sigma} \times \hat{T}_\sigma.$$

Lemma 1.1 *Let X_Σ be the toric variety associated to the complete fan Σ in $\mathbf{R}^d$. There exists a finite CW-cell decomposition of X_Σ such that $p^{-1}(\hat{\sigma})$ is a CW-subcomplex for all cones $\sigma \in \Sigma$.*

Proof For each cone $\sigma \in \Sigma$ let $\bar{\pi}_\sigma : T^d \to \sigma^\perp/\pi_\sigma(\mathbf{Z}^d)$ be the map induced by the orthogonal projection π_σ of $\mathbf{R}^d$ onto the subspace

$$\sigma^\perp = \{x \in \mathbf{R}^d \mid x \cdot y = 0 \quad \forall y \in \sigma\}.$$

Since $\ker \bar{\pi}_\sigma = T_\sigma$ there is an isomorphism $\hat{\pi}_\sigma : \hat{T}_\sigma \to \sigma^\perp/\pi_\sigma(\mathbf{Z}^d)$. Thus a finite CW-cell decomposition of the torus $\hat{T}_\sigma$ is equivalent to a CW-cell decomposition of the subspace $\sigma^\perp$ which is periodic with respect to the group $\pi_\sigma(\mathbf{Z}^d)$ such that there are only finitely many classes of periodic cells. Such a decomposition of $\sigma^\perp$ can be defined as follows.

For a non-zero vector $v \in \sigma^\perp$ let $\mathcal{H}(v) = \{x \in \sigma^\perp \mid x \cdot v \in \mathbf{Z}\}$ be the band of parallel hyperplanes generated by v. If $V \subset \sigma^\perp$ is a finite set of non-zero vectors which span $\sigma^\perp$, the closures of the components of $\sigma^\perp \setminus \bigcup_{v \in V} \mathcal{H}(v)$ are convex polytopes which, together with their faces, form a polyhedral complex $\mathcal{P}(V)$ with support $\sigma^\perp$. If in addition the vectors $v \in V$ are integral, the corresponding hyperplane bands $\mathcal{H}(v)$ are $\pi_\sigma(\mathbf{Z}^d)$-periodic and hence the polyhedral complex $\mathcal{P}(V)$ too. Note that, if $V' \subset \sigma^\perp$ is another finite set of non-zero vectors which contains V, the complex $\mathcal{P}(V')$ is a refinement of $\mathcal{P}(V)$.

For each pair of incident cones $\tau \subset \sigma$ we now have the following commutative diagram

$$\begin{array}{ccc} \hat{T}_\tau & \xrightarrow[\cong]{\hat{\pi}_\tau} & \tau^\perp/\pi_\tau(\mathbf{Z}^d) \\ {\scriptstyle p_{\tau,\sigma}}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{\pi}_{\tau,\sigma}} \\ \hat{T}_\sigma & \xrightarrow[\cong]{\hat{\pi}_\sigma} & \sigma^\perp/\pi_\sigma(\mathbf{Z}^d) \end{array}$$

where $\bar{\pi}_{\tau,\sigma}$ is the map induced by the orthogonal projection $\pi_{\tau,\sigma} : \tau^\perp \to \sigma^\perp$. Therefore, in order to obtain finite CW-cell decompositions of the tori $\hat{T}_\sigma$ such that the projection maps $p_{\tau,\sigma}$ are cellular, we have to define for each cone $\sigma \in \Sigma$ a finite set $V_\sigma \subset \sigma^\perp$ of non-zero integral vectors which span $\sigma^\perp$, such that the projection maps $\pi_{\tau,\sigma}$ are cellular with respect to the corresponding polyhedral complexes $\mathcal{P}(V_\sigma)$. Since the composition of two cellular maps is cellular and $\pi_{\theta,\sigma} = \pi_{\tau,\sigma} \circ \pi_{\theta,\tau}$ for $\theta \subset \tau \subset \sigma$, it is sufficient to consider pairs of incident cones $\tau \subset \sigma$ whose dimensions differ by 1.

We proceed inductively and assume that we have already defined the sets V_τ for all cones $\tau \in \Sigma$ of dimension k. (For the cone $\{0\}$ we may start with the set of coordinate vectors which generates the standard CW-cell decomposition of the torus T^d.) Let σ be a $(k+1)$-dimensional cone and $\tau \subset \sigma$ a k-dimensional face. For every pair of linearly independent vectors $v_1, v_2 \in V_\tau$ which not both lie in $\sigma^\perp$, the orthogonal projection of the intersection $\mathcal{H}(v_1) \cap \mathcal{H}(v_2)$ onto $\sigma^\perp$ yields a band of parallel hyperplanes in $\sigma^\perp$ generated by an integral vector $v' \in \sigma^\perp$. (In fact v' is a linear combination of the vectors v_1 and v_2 with coprime integral coefficients.) Let $V_{\tau,\sigma}$ be the set of all vectors v' obtained in this way. Then the projection map $\pi_{\tau,\sigma}$ is cellular with respect to the complexes $\mathcal{P}(V_\tau)$ and $\mathcal{P}(V_{\tau,\sigma})$. Thus, if we define V_σ to be the union of the sets $V_{\tau,\sigma}$ where τ ranges over the faces of σ of codimension 1, $\mathcal{P}(V_\sigma)$ is the common refinement of the complexes $\mathcal{P}(V_{\tau,\sigma})$ and hence all the projection maps $\pi_{\tau,\sigma}$ are cellular, as required.

That $p^{-1}(\hat{\sigma})$ is a CW-subcomplex of X_Σ follows from the fact that $p^{-1}(\hat{\sigma})$ is a closed subspace which consists of cells of X_Σ. □

Remark Though the CW-cell decomposition of X_Σ is finite, it seems hard to explicitly compute the corresponding cellular homology of X_Σ, except in the low dimensional case (see section 3).

2 A cohomology spectral sequence for toric varieties

Let $p : E \to B$ be a fibration over a pathwise and simply connected base space B. Then, by the Leray-Serre theorem, there exists a spectral sequence which converges to the cohomology of the total space E and whose E_2-term is isomorphic to the cohomology of B with coefficients in the cohomology of a fibre $p^{-1}(b),\ b \in B$. In the case of a toric variety X_Σ, the projection $p : X_\Sigma \to B^d$ is not a fibration since the tori $\hat{T}_\sigma \cong p^{-1}(x),\ x \in \operatorname{int} \hat{\sigma} \subset B^d$, do not all have the same cohomology. Nevertheless there exists a spectral sequence similar to that of a fibration. But in order to describe its E_2-term, we have to introduce the notion of a functor on the fan Σ and its associated cohomology which represents the different cohomology groups of the tori $\hat{T}_\sigma$ and the transition maps between them. Since the cohomology of the torus $\hat{T}_\sigma$ is isomorphic to the exterior algebra on the subgroup of $\mathbf{Z}^d$ orthogonal to the cone σ, we obtain an alternative description of the E_2-term.

2.1 Functors on fans and associated cohomology

A fan Σ can be regarded as a category whose objects are the cones of Σ and whose morphisms are the inclusions between them. Thus, a contravariant functor F from a fan Σ to a category $\mathcal{C}$ is a map which associates to each cone $\sigma \in \Sigma$ an object $F(\sigma) \in \mathcal{C}$ and to each inclusion $\tau \subset \sigma$ a morphism $F(\tau, \sigma) : F(\sigma) \to F(\tau)$, such that $F(\sigma, \sigma) = \mathrm{id}_{F(\sigma)}$ and $F(\theta, \sigma) = F(\theta, \tau) \circ F(\tau, \sigma)$ for $\theta \subset \tau \subset \sigma$.

Example 2.1 Let H^*_Σ be the functor which associates to each cone $\sigma \in \Sigma$ the cohomology ring $H^*(\hat{T}_\sigma)$ of the torus $\hat{T}_\sigma$, and to each inclusion $\tau \subset \sigma$ the homomorphism $p^*_{\tau,\sigma} : H^*(\hat{T}_\sigma) \to H^*(\hat{T}_\tau)$ induced by the canonical projection map $p_{\tau,\sigma} : \hat{T}_\tau \to \hat{T}_\sigma$.

Example 2.2 Let $\bigwedge^*_\Sigma$ be the functor which associates to each cone $\sigma \in \Sigma$ the exterior algebra $\bigwedge^*(\sigma^\vee)$ on the subgroup $\sigma^\vee = \sigma^\perp \cap \mathbf{Z}^d$ of $\mathbf{Z}^d$, and to each inclusion $\tau \subset \sigma$ the induced inclusion map $i^\wedge_{\tau,\sigma} : \bigwedge^*(\sigma^\vee) \to \bigwedge^*(\tau^\vee)$.

For each cone $\sigma \in \Sigma$ we define its orientation group O_σ to be the free abelian group generated by the pairs $(\sigma, o), (\sigma, \bar{o})$ modulo the relation $(\sigma, o) + (\sigma, \bar{o}) = 0$ where o and $\bar{o}$ are the two orientations of σ. (Note that $O_\sigma \cong \mathbf{Z}$.) For a face $\tau \subset \sigma$ of codimension 1 let $\omega_{\tau,\sigma} : O_\sigma \to O_\tau$ denote the homomorphism given by $[(\sigma, o)] \mapsto [(\tau, o')]$ where o' is the orientation of τ induced by o.

Definition 2.1 Let Σ be a complete fan in $\mathbf{R}^d$ and F a contravariant functor from Σ to the category of abelian groups. For $0 \leq s \leq d$ we set

$$C^s(\Sigma, F) = \bigoplus_{\sigma \in \Sigma^{(d-s)}} O_\sigma \otimes F(\sigma).$$

Then a coboundary map $\delta^s : C^s(\Sigma, F) \to C^{s+1}(\Sigma, F)$ is defined by

$$\delta^s|_{O_\sigma \otimes F(\sigma)} = \sum_{\substack{\tau \in \Sigma^{(d-s-1)} \\ \tau \subset \sigma}} \omega_{\tau,\sigma} \otimes F(\tau, \sigma).$$

We call $C^*(\Sigma, F) = (C^s(\Sigma, F), \delta^s)_{s \in \mathbf{Z}}$ the cochain complex of the fan Σ associated to the functor F. Let $H^*(\Sigma, F)$ denote the corresponding homology.

Example 2.3 Let G_Σ be the constant functor which associates to each cone $\sigma \in \Sigma$ the fixed abelian group G and to each inclusion $\tau \subset \sigma$ the identity map on G. Then the associated cochain complex $C^*(\Sigma, G_\Sigma)$ is isomorphic to the augmented cellular chain complex of the spherical complex $\mathcal{C} = \{\sigma \cap S^{d-1} | \ \sigma \in \Sigma, \sigma \neq \{0\}\}$ with coefficients in G. Therefore $H^0(\Sigma, G_\Sigma) \cong G$ and $H^s(\Sigma, G_\Sigma) = 0$ for $s \neq 0$.

The cochain complex $C^*(\Sigma, F)$ is natural with respect to the functor F in the following sense.

Lemma 2.1 *Let F_1, F_2 be two contravariant functors from the fan Σ to the category of abelian groups. If there exists a natural transformation Φ from F_1 to F_2, then Φ induces a chain map $f : C^*(\Sigma, F_1) \to C^*(\Sigma, F_2)$. Furthermore, if Φ is a natural equivalence, then f is an isomorphism and hence $H^*(\Sigma, F_1) \cong H^*(\Sigma, F_2)$.*

Proof Let Φ be a natural transformation from F_1 to F_2, i.e. a map which associates to each cone $\sigma \in \Sigma$ a homomorphism $\Phi_\sigma : F_1(\sigma) \to F_2(\sigma)$ such that $\Phi_\tau \circ F_1(\tau, \sigma) = F_2(\tau, \sigma) \circ \Phi_\sigma$ for every inclusion $\tau \subset \sigma$. For $0 \leq s \leq d$ we define a homomorphism $f^s : C^s(\Sigma, F_1) \to C^s(\Sigma, F_2)$ by

$$f^s|_{O_\sigma \otimes F_1(\sigma)} = \sum_{\sigma \in \Sigma^{(d-s)}} \mathrm{id}_{O_\sigma} \otimes \Phi_\sigma.$$

It follows that $f^{s+1} \circ \delta_1^s = \delta_2^s \circ f^s$ for all s, hence $f = (f^s)_{s \in \mathbf{Z}}$ is a chain map. If Φ is a natural equivalence, i.e. the homomorphisms Φ_σ are isomorphisms, the maps f^s are isomorphisms, too. □

Let $\sigma \in \Sigma$ be a k-dimensional cone. Then $\hat{T}_\sigma$ is a torus of dimension $d-k$ and $\sigma^\vee$ is a free abelian group of rank $d-k$, hence the cohomology ring $H^*(\hat{T}_\sigma)$ is isomorphic to the exterior algebra $\bigwedge^*(\sigma^\vee)$ (see example B.2). Moreover, the corresponding isomorphisms can be chosen simultaneously for all cones $\sigma \in \Sigma$ such that the following lemma holds.

Lemma 2.2 *Let Σ be a complete fan in $\mathbf{R}^d$. There is a natural equivalence between the functors H^*_Σ and $\bigwedge^*_\Sigma$.*

Proof For each cone $\sigma \in \Sigma$ we have to find an isomorphism $\Phi_\sigma : H^*(\hat{T}_\sigma) \to \bigwedge^*(\sigma^\vee)$ such that for every inclusion $\tau \subset \sigma$ the following diagram is commutative:

$$\begin{array}{ccc} H^*(\hat{T}_\sigma) & \xrightarrow{p^*_{\tau,\sigma}} & H^*(\hat{T}_\tau) \\ \Phi_\sigma \Big\downarrow \cong & & \Phi_\tau \Big\downarrow \cong \\ \bigwedge^*(\sigma^\vee) & \xrightarrow{i^{\wedge}_{\tau,\sigma}} & \bigwedge^*(\tau^\vee) \end{array}$$

We first define the isomorphism $\Phi = \Phi_{\{0\}} : H^*(T^d) \to \bigwedge^*(\mathbf{Z}^d)$. For a vector $v \in \mathbf{Z}^d$ let $c_v \in H_1(T^d)$ be the homology class represented by the cycle which is obtained by projecting the segment $\{tv|\ 0 \le t \le 1\} \subset \mathbf{R}^d$ onto the torus T^d. The map $v \mapsto c_v$ defines an isomorphism from $\mathbf{Z}^d$ to $H_1(T^d)$ and since $H^1(T^d) \cong \mathrm{Hom}(H_1(T^d), \mathbf{Z})$, there is a vector $\Phi(\alpha) \in \mathbf{Z}^d$ for each cohomology class $\alpha \in H^1(T^d)$ such that

$$\alpha(c_v) = \Phi(\alpha) \cdot v \qquad \forall\, v \in \mathbf{Z}^d.$$

Let $\Phi : H^*(T^d) \to \bigwedge^*(\mathbf{Z}^d)$ be the induced ring isomorphism. For a cone $\sigma \in \Sigma$, $\sigma \neq \{0\}$, we now define $\Phi_\sigma : H^*(\hat{T}_\sigma) \to \bigwedge^*(\mathbf{Z}^d)$ to be the composition of the ring homomorphism $p^*_\sigma : H^*(\hat{T}_\sigma) \to H^*(T^d)$ induced by the canonical projection $p_\sigma : T^d \to \hat{T}_\sigma$, followed by Φ. We show that Φ_σ is injective and that $\mathrm{im}\, \Phi_\sigma = \bigwedge^*(\sigma^\vee)$. The commutativity of the diagram will then follow from the fact that $p^*_\sigma = p^*_\tau \circ p^*_{\tau,\sigma}$ for $\tau \subset \sigma$.

(i) Since the homology of the torus $\hat{T}_\sigma$ is free and of finite rank, it follows from $T^d \cong T_\sigma \oplus \hat{T}_\sigma$ that $H^*(T^d) \cong H^*(T_\sigma) \otimes H^*(\hat{T}_\sigma)$ by the Künneth theorem. Under this isomorphism the homomorphism $p^*_\sigma : H^*(\hat{T}_\sigma) \to H^*(T^d)$ is given by $\alpha \mapsto 1 \otimes \alpha$ where $1 \in H^*(T_\sigma)$ is the unit element. Therefore p^*_σ is injective and $\Phi_\sigma = \Phi \circ p^*_\sigma$ as well.

(ii) Since Φ_σ is a ring homomorphism and $H^*(\hat{T}_\sigma)$ is generated by $H^1(\hat{T}_\sigma)$ just as $\bigwedge^*(\sigma^\vee)$ is generated by $\bigwedge^1(\sigma^\vee) \cong \sigma^\vee$, it is sufficient to show that $\mathrm{im}\, \Phi_\sigma|_{H^1(T^d)} = \sigma^\vee$. From $\ker p_\sigma = T_\sigma$ it follows that $\ker p_{\sigma *}|_{H_1(T^d)} = \{c_v \in H_1(T^d)|\ v \in \mathrm{span}\, \sigma \cap \mathbf{Z}^d\}$. Thus for $\alpha \in H^1(T^d)$ we have

$$\Phi(p^*_\sigma \alpha) \cdot v = (p^*_\sigma \alpha)(c_v) = \alpha(p_{\sigma *}(c_v)) = 0 \qquad \forall\, v \in \mathrm{span}\, \sigma \cap \mathbf{Z}^d$$

and hence $\Phi_\sigma(\alpha) = \Phi(p^*_\sigma \alpha) \in \sigma^\vee$. Reversely, if $\Phi(\beta) \in \sigma^\vee$ for some $\beta \in H^1(T^d)$, then $\beta(c_v) = 0$ for all $v \in \mathrm{span}\, \sigma \cap \mathbf{Z}^d$ and hence $\ker p_{\sigma *}|_{H_1(T^d)} \subset \ker \beta$. Thus there exists a homomorphism $\alpha : H_1(\hat{T}_\sigma) \to \mathbf{Z}$ such that $\beta = \alpha \circ p_{\sigma *}|_{H_1(T^d)} = p^*_\sigma(\alpha)$. □

Remark Since $H^*(\hat{T}_\sigma)$ and $\bigwedge^*(\sigma^\vee)$ both are graded rings and the isomorphism Φ_σ respects their gradations, there also exists a natural equivalence for each $0 \leq t \leq d$ between the functors H^t_Σ and $\bigwedge^t_\Sigma$ which associate to each cone $\sigma \in \Sigma$ the groups $H^t(\hat{T}_\sigma)$ and $\bigwedge^t(\sigma^\vee)$, respectively.

2.2 The spectral sequence

Let X_Σ be the toric variety associated to the complete fan Σ in $\mathbf{R}^d$. By taking the preimages $X_s = p^{-1}(B_s)$ where B_s is the union of the cells $\operatorname{int}\hat{\sigma} \subset B^d$ of dimension $\leq s$, we obtain a natural filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_d = X_\Sigma.$$

By lemma 1.1 the sets X_s are CW-subcomplexes of X_Σ, hence there is an associated spectral sequence $(E_r, d_r)_{r\geq 1}$ with initial term $E_1^{s,t} \cong H^{s+t}(X_s, X_{s-1})$ and differential $d_1 : E_1^{s,t} \to E_1^{s+1,t}$ corresponding to the connecting homomorphism in the long exact cohomology sequence of the triple (X_{s+1}, X_s, X_{s-1}) (see example A.1). Since the filtration of X_Σ is finite the spectral sequence degenerates, i.e. eventually becomes constant, and the limit term E_∞ is isomorphic to the bigraded module associated to the filtration F of $H^*(X_\Sigma)$ defined by

$$F^s H^*(X_\Sigma) = \ker[H^*(X_\Sigma) \to H^*(X_{s-1})].$$

By computing the E_1-term and its differential d_1 we will prove the following

Theorem 2.1 *Let X_Σ be the toric variety associated to the complete fan Σ in $\mathbf{R}^d$. There exists a spectral sequence $(E_r, d_r)_{r\geq 1}$ with $E_2^{s,t} \cong H^s(\Sigma, \bigwedge^t_\Sigma)$ which converges to the cohomology of X_Σ.*

Proof In order to compute the E_1-term, we may write the pair (X_s, X_{s-1}) as the union of the pairs $(p^{-1}(\hat{\sigma}), p^{-1}(\partial\hat{\sigma}))$ where σ ranges over the $(d-s)$-dimensional cones of Σ. Since the pairs $(p^{-1}(\hat{\sigma}), p^{-1}(\partial\hat{\sigma}))$ are CW-subcomplexes of (X_s, X_{s-1}) and the intersection of every two of them has trivial cohomology (as a pair consisting of equal components), it follows by induction from the Mayer-Vietoris sequence that

$$H^{s+t}(X_s, X_{s-1}) \cong \bigoplus_{\sigma \in \Sigma^{(d-s)}} H^{s+t}(p^{-1}(\hat{\sigma}), p^{-1}(\partial\hat{\sigma})).$$

Under this isomorphism the differential $d_1 : H^{s+t}(X_s, X_{s-1}) \to H^{s+t+1}(X_{s+1}, X_s)$ induces for each $(d-s)$-dimensional cone $\sigma \in \Sigma$ and each face $\tau \subset \sigma$ of codimension 1 a map $H^{s+t}(p^{-1}(\hat{\sigma}), p^{-1}(\partial\hat{\sigma})) \to H^{s+t+1}(p^{-1}(\hat{\tau}), p^{-1}(\partial\hat{\tau}))$ which is equal to the composition $(i^*)^{-1} \circ \delta^*$ in the first row of the following diagram:

$$
\begin{array}{ccccccc}
H^{s+t}(p^{-1}(\hat\sigma),p^{-1}(\partial\hat\sigma)) & & \xleftarrow[\cong]{\quad i^* \quad} & & H^{s+t}(p^{-1}(\partial\hat\tau),p^{-1}(\partial\hat\tau\setminus\operatorname{int}\hat\sigma)) & \xrightarrow{\delta^*} & H^{s+t+1}(p^{-1}(\hat\tau),p^{-1}(\partial\hat\tau)) \\
{\scriptstyle f_\sigma^*}\downarrow{\scriptstyle\cong} & & (\mathbf{1}) & & \downarrow{\scriptstyle f_\tau^*|_{p^{-1}(\partial\hat\tau)}} & (\mathbf{2}) & {\scriptstyle\cong}\downarrow{\scriptstyle f_\tau^*} \\
H^{s+t}(\hat\sigma\times\hat T_\sigma,\partial\hat\sigma\times\hat T_\sigma) & \xrightarrow{(\mathrm{id}\times p_{\tau,\sigma})^*} & H^{s+t}(\hat\sigma\times\hat T_\tau,\partial\hat\sigma\times\hat T_\tau) & \xleftarrow[\cong]{i^*} & H^{s+t}(\partial\hat\tau\times\hat T_\tau,(\partial\hat\tau\setminus\operatorname{int}\hat\sigma)\times\hat T_\tau) & \xrightarrow{\delta^*} & H^{s+t+1}(\hat\tau\times\hat T_\tau,\partial\hat\tau\times\hat T_\tau) \\
{\scriptstyle\lambda_\sigma}\uparrow{\scriptstyle\cong} & (\mathbf{3}) & {\scriptstyle\lambda'}\uparrow{\scriptstyle\cong} & (\mathbf{4}) & {\scriptstyle\cong}\uparrow{\scriptstyle\lambda''} & (\mathbf{5}) & {\scriptstyle\cong}\uparrow{\scriptstyle\lambda_\tau} \\
H^s(\hat\sigma,\partial\hat\sigma)\otimes H^t(\hat T_\sigma) & \xrightarrow{\mathrm{id}\otimes p_{\tau,\sigma}^*} & H^s(\hat\sigma,\partial\hat\sigma)\otimes H^t(\hat T_\tau) & \xleftarrow[\cong]{i^*\otimes\mathrm{id}} & H^s(\partial\hat\tau,\partial\hat\tau\setminus\operatorname{int}\hat\sigma)\otimes H^t(\hat T_\tau) & \xrightarrow{\delta^*\otimes\mathrm{id}} & H^{s+1}(\hat\tau,\partial\hat\tau)\otimes H^t(\hat T_\tau) \\
{\scriptstyle\mu_\sigma\otimes\mathrm{id}}\uparrow{\scriptstyle\cong} & (\mathbf{6}) & {\scriptstyle\mu_\sigma\otimes\mathrm{id}}\uparrow{\scriptstyle\cong} & & (\mathbf{7}) & & {\scriptstyle\cong}\uparrow{\scriptstyle\mu_\tau\otimes\mathrm{id}} \\
O_\sigma\otimes H^t(\hat T_\sigma) & \xrightarrow{\mathrm{id}\otimes p_{\tau,\sigma}^*} & O_\sigma\otimes H^t(\hat T_\tau) & & \xrightarrow{\qquad\omega_{\tau,\sigma}\otimes\mathrm{id}\qquad} & & O_\tau\otimes H^t(\hat T_\tau)
\end{array}
$$

In the diagram on the previous page δ^* always denotes the connecting homomorphism in the long exact cohomology sequence of the corresponding triple of spaces and i^* the homomorphism induced by the inclusion of the corresponding pair of spaces. Since these inclusions are all of the form $(A, A \cap B) \subset (A \cup B, B)$ where A and B are CW-subcomplexes of $A \cup B$, they induce isomorphisms in cohomology by excision. We now describe the vertical isomorphisms and show the commutativity of the diagram, level by level.

(i) For each cone $\sigma \in \Sigma$ we define a map $f_\sigma : (\hat{\sigma} \times \hat{T}_\sigma, \partial\hat{\sigma} \times \hat{T}_\sigma) \to (p^{-1}(\hat{\sigma}), p^{-1}(\partial\hat{\sigma}))$ by $f_\sigma(x,t) = (x, p_{\sigma,\sigma'}(t))$ where $\sigma' \in \Sigma$ is the unique cone for which $x \in \operatorname{int} \hat{\sigma}'$. Since f_σ is the identity map on $\operatorname{int} \hat{\sigma} \times \hat{T}_\sigma$, it is a relative homeomorphism between finite CW-complexes and hence induces an isomorphism in cohomology. The commutativity of square (1) follows from the commutativity of the following diagram

$$\begin{array}{ccccc}
p^{-1}(\hat{\sigma}) & & \hookrightarrow & & p^{-1}(\partial\hat{\tau}) \\
\uparrow{\scriptstyle f_\sigma} & & & & \uparrow{\scriptstyle f_\tau|_{\partial\hat{\tau}\times\hat{T}_\tau}} \\
\hat{\sigma} \times \hat{T}_\sigma & \xleftarrow{\mathrm{id}\times p_{\tau,\sigma}} & \hat{\sigma} \times \hat{T}_\tau & \hookrightarrow & \partial\hat{\tau} \times \hat{T}_\tau
\end{array}$$

and square (2) is commutative because f_τ induces a chain map between the corresponding chain complexes.

(ii) Since the homology of the torus $\hat{T}_\sigma$ is free and of finite rank, there exists by the Künneth theorem an isomorphism $\lambda_\sigma : H^*(\hat{\sigma}, \partial\hat{\sigma}) \otimes H^*(\hat{T}_\sigma) \to H^*(\hat{\sigma} \times \hat{T}_\sigma, \partial\hat{\sigma} \times \hat{T}_\sigma)$ given by $\alpha \otimes \beta \mapsto \alpha \times \beta$. The isomorphisms λ' and λ'' are analogously defined. Since the cohomology cross product is natural (see [9], 5.6.2), the squares (3) and (4) are commutative and the commutativity of square (5) follows from [9], 5.6.6.

(iii) If we fix an orientation of the sphere $S^{d-1} \subset \mathbf{R}^d$, then every orientation o of the cone $\sigma \in \Sigma$ induces a unique orientation of the cell $\hat{\sigma} \subset S^{d-1}$, which is represented by a generating element $\alpha \in H^s(\hat{\sigma}, \partial\hat{\sigma})$ where $s = \dim \hat{\sigma}$. The map $o \mapsto \alpha$ induces an isomorphism $\mu_\sigma : O_\sigma \to H^s(\hat{\sigma}, \partial\hat{\sigma})$ such that square (7) is commutative. The commutativity of square (6) is trivial.

By taking the composition of the isomorphisms f_σ^*, $(\lambda_\sigma)^{-1}$ and $(\mu_\sigma \otimes \mathrm{id})^{-1}$ we see that

$$E_1^{s,t} \cong H^{s+t}(X_s, X_{s-1}) \cong \bigoplus_{\sigma \in \Sigma^{(d-s)}} O_\sigma \otimes H^t(\hat{T}_\sigma) = C^s(\Sigma, H^t_\Sigma)$$

and that the differential $d_1 : E_1^{s,t} \to E_1^{s+1,t}$ corresponds under this isomorphism to the coboundary map δ^s of the cochain complex $C^*(\Sigma, H^t_\Sigma)$. Therefore $E_2^{s,t} \cong H^s(\Sigma, H^t_\Sigma)$ and since there is a natural equivalence between the functors H^t_Σ and $\bigwedge^t_\Sigma$ (see lemma 2.2 and the remark after), the proof is completed. □

Corollary 2.1 *$E_1^{s,t}$ is a free abelian group of rank $a_{d-s}\binom{s}{t}$ for $0 \le s \le d,\ 0 \le t \le s$, and $E_1^{s,t} \cong 0$ otherwise. In particular (E_r, d_r) is a first quadrant spectral sequence.*

2.3 Applications

We show how the spectral sequence of theorem 2.1 can be used to compute the Euler characteristic of a toric variety X_Σ and the low and high dimensional cohomology groups. Recall that the limit term of the spectral sequence is related to the cohomology of X_Σ by

$$\begin{aligned} E_\infty^{s,t} &\cong F^s H^{s+t}(X_\Sigma)/F^{s+1}H^{s+t}(X_\Sigma) \\ &\cong \ker[H^{s+t}(X_\Sigma) \to H^{s+t}(X_{s-1})]/\ker[H^{s+t}(X_\Sigma) \to H^{s+t}(X_s)] \end{aligned}$$

Theorem 2.2 *The Euler characteristic $\chi(X_\Sigma)$ of a d-dimensional toric variety X_Σ is equal to the number a_d of d-dimensional cones in the fan Σ.*

Proof For every term E_r of the spectral sequence we define its Euler characteristic by $\chi(E_r) = \sum_{s,t}(-1)^{s+t}\operatorname{rk} E_r^{s,t}$. By corollary 2.1 we have

$$\chi(E_1) = \textstyle\sum_{s=0}^{d}\sum_{t=0}^{s}(-1)^{s+t}a_{d-s}\binom{s}{t} = a_d.$$

From [9], 4.3.14, it follows that $\chi(E_{r+1}) = \chi(H(E_r)) = \chi(E_r)$ and hence we obtain by induction $\chi(E_r) = a_d$ for all r. Since the spectral sequence degenerates, i.e. $E_\infty \cong E_r$ for some r, we also have $\chi(E_\infty) = a_d$. On the other hand, it follows from

$$\textstyle\sum_{s+t=q}\operatorname{rk} E_\infty^{s,t} = \sum_{s=0}^{d}\operatorname{rk}(F^sH^q(X_\Sigma)/F^{s+1}H^q(X_\Sigma)) = \operatorname{rk} H^q(X_\Sigma)$$

that $\chi(E_\infty) = \chi(X_\Sigma)$. □

Theorem 2.3 *Let X_Σ be a d-dimensional toric variety. Then the low dimensional cohomology groups of X_Σ are given by*

$$H^q(X_\Sigma) \cong \begin{cases} \mathbf{Z} & q=0 \\ 0 & q=1 \\ \ker[\delta^1 : C^1(\Sigma, \textstyle\bigwedge_\Sigma^1) \to C^2(\Sigma, \textstyle\bigwedge_\Sigma^1)] & q=2 \\ H^2(\Sigma, \textstyle\bigwedge_\Sigma^1) & q=3 \end{cases}$$

and the high dimensional groups by

$$H^q(X_\Sigma) \cong \begin{cases} \mathbf{Z}^{a_1-d} \oplus \textstyle\bigwedge^{d-2}(\mathbf{Z}^d)/\sum_{\tau\in\Sigma^{(1)}}\bigwedge^{d-2}(\tau^\vee) & q=2d-2 \\ \textstyle\bigwedge^{d-1}(\mathbf{Z}^d)/\sum_{\tau\in\Sigma^{(1)}}\bigwedge^{d-1}(\tau^\vee) & q=2d-1 \\ \mathbf{Z} & q=2d \end{cases}$$

Proof Since $\bigwedge^0(\sigma^\vee) \cong \mathbf{Z}$ for all cones $\sigma \in \Sigma$, it follows from example 2.3 that

$$E_2^{s,0} \cong H^s(\Sigma, \textstyle\bigwedge_\Sigma^0) \cong \begin{cases} \mathbf{Z} & s = 0 \\ 0 & s \neq 0 \end{cases}$$

By corollary 2.1 we know in addition that $E_2^{s,t} \cong 0$ for $t < 0$, $s > d$ and $t > s$. This implies that for $s + t \leq 3$ and $s + t \geq 2d - 2$ the groups $E_r^{s,t}$, $r \geq 2$, are constant, for the differentials from and to these groups are all trivial, and hence $E_\infty^{s,t} \cong E_2^{s,t}$. On the other hand, the groups $E_\infty^{s,t}$ with $s + t = q$ determine the qth cohomology group of the toric variety X_Σ up to group extensions. In particular, if there is only one non-trivial group $E_\infty^{s,t}$ with $s + t = q$, this group must coincide with $H^q(X_\Sigma)$. This is the case for $q = 0, 2, 3, 2d - 1, 2d$ where the respective non-trivial groups are $E_2^{0,0}, E_2^{1,1}, E_2^{2,1}, E_2^{d,d-1}$ and $E_2^{d,d}$. For $q = 1$ all the groups $E_2^{s,t}$ with $s + t = q$ are trivial and hence $H^1(X_\Sigma)$ is trivial, too. For $q = 2d - 2$ there are two non-trivial groups $E_2^{s,t}$ with $s + t = q$ and we first obtain the isomorphism $E_2^{d-1,d-1} \cong H^{2d-2}(X_\Sigma)/E_2^{d,d-2}$. But since $E_2^{d-1,d-1} \cong \ker[d_1 : E_1^{d-1,d-1} \to E_1^{d,d-1}]$ is free, $H^{2d-2}(X_\Sigma)$ splits into the direct sum $E_2^{d-1,d-1} \oplus E_2^{d,d-2}$. Finally the proof can be completed by interpreting the corresponding groups $E_2^{s,t} \cong H^s(\Sigma, \bigwedge_\Sigma^t)$. □

Remarks (i) The formula for $H^2(X_\Sigma)$ is a generalization of a result by M. Eikelberg (see [2], 7.3 Folgerung, 7.7 Satz), since it also holds for toric varieties whose underlying fans are not polytopal.
(ii) The high dimensional cohomology groups do not depend on the combinatorial structure of the fan Σ but on its generating vectors.
(iii) All the groups which occur in the theorem can easily be calculated within multilinear algebra. Thus the theorem enables us in particular to determine the cohomology of a 3-dimensional toric variety.

Example 2.4 Let Σ be the fan generated by the cube $[-1, 1]^3 \subset \mathbf{R}^3$ (see example 1.1) and Σ' the fan obtained from Σ by replacing the generating vector $v = (1, 1, 1)$ with the vector $v' = (2, 1, 1)$. Then the associated toric variety $X_{\Sigma'}$ has the following cohomology groups:

$$H^q(X_{\Sigma'}) \cong \begin{cases} 0 & q = 2 \\ \mathbf{Z} & q = 3 \\ \mathbf{Z}^5 & q = 4 \\ \mathbf{Z}/2\mathbf{Z} & q = 5 \end{cases}$$

This example shows that the second cohomology group of a toric variety can vanish, if the underlying fan is not polytopal.

3 Classification of smooth 2-dimensional toric varieties

In this section we use the CW-cell decomposition of section 1 to explicitly calculate the intersection form of a smooth 2-dimensional toric variety X_Σ. Applying Freedman's characterisation of 4-manifolds by their intersection form, we conclude that X_Σ is either homeomorphic to the complex projective plane $\mathbf{C}P^2$, the product of spheres $S^2 \times S^2$ or the connected sum of $\mathbf{C}P^2$ with a finite number of $-\mathbf{C}P^2$. This result could also be deduced from a theorem of Oda ([8], theorem 8.2) which uses techniques from algebraic geometry and combinatorial properties of regular fans in $\mathbf{R}^2$.

3.1 Cellular homology

Let X_Σ be a 2-dimensional (not necessarily smooth) toric variety associated to the complete fan Σ in $\mathbf{R}^2$. We assume that the 1-dimensional cones $\tau_1, \ldots, \tau_n$ of Σ are numbered counterclockwise and that the cone τ_i is generated by the primitive vector $v_i = (v_{i1}, v_{i2}) \in \mathbf{Z}^2$ (i.e. v_{i1} and v_{i2} are coprime integers). Furthermore let σ_i be the 2-dimensional cone which is generated by the vectors v_i and v_{i+1}. (The indexing from 1 to n is always meant to be cyclic.) For convenience we may represent the dual complex $\hat{\Sigma}$ as the face complex of a polygon P, which we hence designate as the dual of the cone $\{0\}$.

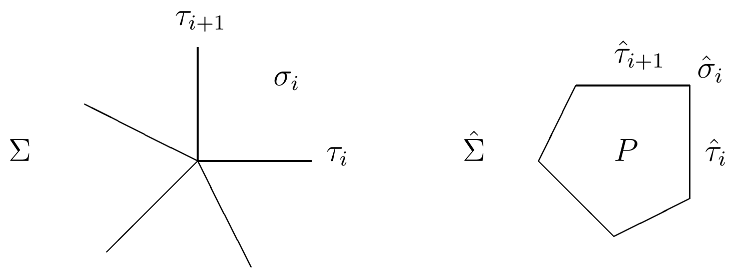

Figure 1: A 2-dimensional fan Σ and its dual complex $\hat{\Sigma}$

We first describe for each cone $\varrho \in \Sigma$ the CW-cell decomposition of the torus $\hat{T}_\varrho$ which we have used in the proof of lemma 1.1. Recall that $\hat{T}_\varrho \cong \varrho^\perp / \pi_\varrho(\mathbf{Z}^2)$ where $\varrho^\perp$ is the subspace of $\mathbf{R}^2$ orthogonal to the cone ϱ and π_ϱ the orthogonal projection of $\mathbf{R}^2$ onto $\varrho^\perp$. Let I denote the interval $\{t \in \mathbf{R} | \ 0 < t < 1\}$. Then the standard CW-cell decomposition of the torus $\hat{T}_{\{0\}} \cong T^2$ consists of the cells c^0, c^1_1, c^1_2 and c^2 which are represented by the subsets $\{(0,0)\}, I \times \{0\}, \{0\} \times I$ and $I \times I$ of $\mathbf{R}^2$, respectively. For $1 \le i \le n$ the 1-dimensional torus $\hat{T}_{\tau_i}$ is decomposed into a 0-cell which we identify with c^0 and the 1-cell c_i^1 which is represented by the segment $Iw_i \subset \tau_i^\perp$, where $w_i \in \mathbf{R}^2$ is the vector orthogonal to v_i such that $\det(v_i, w_i) = 1$. Finally, the 0-dimensional tori $\hat{T}_{\sigma_i}$, $1 \leq i \leq n$, consist of a single 0-cell which we identify again with c^0.

The product cells $\operatorname{int}\hat{\varrho} \times c$ now form a CW-cell decomposition of the toric variety X_Σ and if we provide them with appropriate orientations, the boundaries of the corresponding cellular chains (for which we use the same notation) are given by

$$\begin{array}{rcl} \partial_0\,(\operatorname{int}\hat{\sigma}_i \times c^0) & = & 0 \\ \partial_1\,(\operatorname{int}\hat{\tau}_i \times c^0) & = & \operatorname{int}\hat{\sigma}_i \times c^0 - \operatorname{int}\hat{\sigma}_{i-1} \times c^0 \\ \partial_2\,(\operatorname{int}\hat{\tau}_i \times c_i^1) & = & 0 \\ \partial_2\,(\operatorname{int} P \times c^0) & = & \sum(\operatorname{int}\hat{\tau}_i \times c^0) \\ \partial_3\,(\operatorname{int} P \times c_1^1) & = & \sum(-v_{i2})(\operatorname{int}\hat{\tau}_i \times c_i^1) \\ \partial_3\,(\operatorname{int} P \times c_2^1) & = & \sum v_{i1}(\operatorname{int}\hat{\tau}_i \times c_i^1) \\ \partial_4\,(\operatorname{int} P \times c^2) & = & 0 \end{array} \qquad (1)$$

Note that the multiplicities of the chain $\operatorname{int}\hat{\tau}_i \times c_i^1$ in the boundaries of the two 3-chains come from the images $\pi_{\tau_i}(1,0) = -v_{i2}w_i$ and $\pi_{\tau_i}(0,1) = v_{i1}w_i$. By evaluating the above boundary maps we finally obtain the following

Lemma 3.1 *Let X_Σ be a 2-dimensional toric variety associated to the complete fan Σ in $\mathbf{R}^2$ whose 1-dimensional cones are generated by the primitive vectors $v_1, \ldots, v_n \in \mathbf{Z}^2$. Then the homology of X_Σ is given by*

$$H_q(X_\Sigma) \cong \begin{cases} \mathbf{Z} & q = 0, 4 \\ \mathbf{Z}^n/(\mathbf{Z}v_1^* + \mathbf{Z}v_2^*) \cong \mathbf{Z}^{n-2} \oplus \mathbf{Z}/m\mathbf{Z} & q = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $v_1^* = (v_{11}, \ldots, v_{n1}), v_2^* = (v_{12}, \ldots, v_{n2})$, and m is the greatest common divisor of the determinants $\det(v_i, v_j)$, $0 \leq i < j \leq n$. In particular $H_2(X_\Sigma) \cong \mathbf{Z}^{n-2}$ if X_Σ is smooth.

3.2 The intersection form

In the following we assume that the 2-dimensional toric variety X_Σ is smooth. Thus by property (ii) of theorem 1.1 we have

$$\det(v_i, v_{i+1}) = 1 \qquad (1 \leq i \leq n) \qquad (2)$$

and in view of property (iii) we may assume in addition that $v_{n-1} = (1,0)$ and $v_n = (0,1)$. By (1) we see that the group of 2-cycles of X_Σ is generated by the chains $\operatorname{int} \hat{\tau}_i \times c_i^1$, $1 \leq i \leq n$, and the corresponding homology classes $z_i \in H_2(X_\Sigma)$ satisfy the following relations:

$$\begin{array}{lcl} v_{11} z_1 + \ldots + v_{n-2,1} z_{n-2} + z_{n-1} & = & 0 \\ v_{12} z_1 + \ldots + v_{n-2,2} z_{n-2} + z_n & = & 0. \end{array} \tag{3}$$

In order to determine the intersection numbers $z_i \cdot z_j$, we first observe that the spheres $p^{-1}(\hat{\tau}_i)$ and $p^{-1}(\hat{\tau}_j)$ which represent the classes z_i and z_j, respectively, do not intersect in X_Σ if their generating cones τ_i and τ_j are not adjacent. Therefore we have

$$z_i \cdot z_j = 0 \qquad (1 < |i-j| < n-1).$$

Second, the spheres $p^{-1}(\hat{\tau}_i)$ and $p^{-1}(\hat{\tau}_{i+1})$ intersect in the unique point which lies over the vertex $\hat{\sigma}_i$ of P. Since $\det(v_i, v_{i+1}) = 1$, it can be seen that this intersection is transversal and hence

$$z_i \cdot z_{i+1} = \pm 1 \qquad (1 \leq i \leq n)$$

where the signs are all equal and only depend on the orientation of X_Σ. In the following we fix them to be $+1$.

Third, by multiplying both relations (3) with z_i and taking a suitable linear combination of the resulting equations, we obtain the self intersection numbers

$$z_i \cdot z_i = -\det(v_{i-1}, v_{i+1}) \qquad (1 \leq i \leq n)$$

where we have also used the smoothness condition (2).

3.3 Characterisation of the intersection form

Having calculated the intersection form of X_Σ, we now characterise it up to isomorphism, i.e. up to a change of basis of $H_2(X_\Sigma)$. By (3) the classes $z_1, \ldots, z_{n-2}$ form a basis of $H_2(X_\Sigma)$, hence the **rank** of the intersection form equals $n-2$.

In order to determine the **signature**, we first have to calculate the principal minors $D_k = \det((z_i \cdot z_j)_{1 \leq i,j \leq k})$ of the intersection matrix. (Henceforth we will not distinguish between the intersection form and its matrix.) From the results of the previous subsection it follows that $D_1 = \det(v_2, v_n)$ and

$$D_k = -\det(v_{k-1}, v_{k+1})\, D_{k-1} - D_{k-2} \qquad (2 \leq k \leq n-2)$$

where we have set $D_0 = 1$. By induction one can easily prove, e.g. by using the Grassmann-Plücker relation in $\mathbf{R}^2$, that

$$D_k = (-1)^{k+1} \det(v_{k+1}, v_n) \qquad (1 \leq k \leq n-2).$$

From this equation we see that if none of the vectors v_k is equal to $-v_n = (0, -1)$, then all the principal minors are non-zero and have alternating signs, except for the unique pair (D_{k-1}, D_k) for which the vectors v_k and v_{k+1} lie on opposite sides of the y-axis. Hence by Jacobi's theorem the signature of the intersection form equals $4-n$. By a rule of Gundenfinger this still holds even if there exists a vector $v_k = (0, -1)$ in which case $D_{k-1} = 0$ (see [4], note 1 on page 304).

If $n = 3$ there is only one possible vector $v_1 = (-1, -1)$ and the intersection form of X_Σ given by the matrix (1) is positive definite. If $n > 3$ the absolute value of the signature of the intersection form of X_Σ is less than its rank, hence the form is indefinite. Thus in order to characterise it, we finally have to determine its **type**.

The even type is only possible if $n = 4$. Indeed, if the intersection form of X_Σ is even, then all the determinants $\det(v_{i-1}, v_{i+1})$, $1 \le i \le n-2$, must be even. Since the vectors v_i are primitive, it follows that they all have one even and one odd coordinate, thus they are contained in the lattice $\Gamma = (1,0) + \mathbf{Z}(1,1) + \mathbf{Z}(-1,1)$. Hence by Pick's formula (see e.g. [5]) the area $A(S)$ of the star-shaped polygon $S = \bigcup_{i=1}^n \operatorname{conv}\{0, v_i, v_{i+1}\}$ is given by

$$\frac{A(S)}{\det \Gamma} = \operatorname{card}(\Gamma \cap \operatorname{int} S) + \frac{1}{2} \operatorname{card}(\Gamma \cap \partial S) - 1$$

where $\det \Gamma$ denotes the determinant of a basis of Γ. But by condition (2) the area $A(S)$ equals $\frac{n}{2}$ and S does not contain any points of Γ other than its vertices. Therefore the equality can hold only if $n = 4$.

In fact, if $n = 4$ and $v_1 = (-1, 0)$, $v_2 = (0, -1)$ the resulting form is even. On the other hand, every odd indefinite intersection form of rank $n \ge 4$ can also be realised, e.g. by setting $v_i = (i-2, -1)$, $1 \le i \le n-2$.

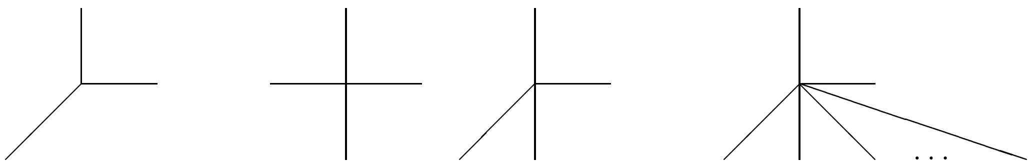

Figure 2: Representative fans with 3, 4 and n generators

Thus we have completely characterised the possible intersection forms of X_Σ and we summarize the results in the following

Theorem 3.1 *A non-singular integral symmetric bilinear form B can be realised as the intersection form of an oriented smooth 2-dimensional toric variety X_Σ if and only if either*
(i) $\operatorname{rank}(B) = 1$ *and* $B = (\pm 1)$*, or*
(ii) $\operatorname{rank}(B) = 2$ *and B is even, or*
(iii) $\operatorname{rank}(B) \ge 2$, $|\operatorname{signature}(B)| = \operatorname{rank}(B) - 2$ *and B is of odd type.*

3.4 Classification

In 1982 Freedman characterised topological 4-manifolds by showing that every non-singular integral symmetric bilinear form can be realised as the intersection form of an oriented closed simply-connected 4-manifold, and that any two such manifolds realising the same form are homeomorphic if the form is even, whereas if the form is odd there are two homeomorphism classes, one with trivial and the other with non-trivial Kirby-Siebenmann obstruction (see [3], theorem 1.5).

In our case it is easy to give representatives of 4-manifolds which realise the intersection forms described in the theorem of the previous subsection. Namely, let us consider the oriented complex projective plane $\mathbf{C}P^2$ which has intersection form $(+1)$. Then $\mathbf{C}P^2$ with the opposite orientation, which we denote by $-\mathbf{C}P^2$, has intersection form (-1). Furthermore, if we take the connected sum of $\mathbf{C}P^2$ with a finite number of copies of $-\mathbf{C}P^2$, we obtain a 4-manifold whose intersection form is the orthogonal sum of $(+1)$ with a finite number of (-1) and hence satisfies condition (iii) of the theorem. Finally, the even indefinite form of rank 2 is the intersection form of the product of spheres $S^2 \times S^2$. All these manifolds are smooth and hence have trivial Kirby-Siebenmann obstruction, and since the same is true for the toric varieties in question, we can state the following

Corollary 3.1 *The homeomorphism classes of a smooth 2-dimensional toric variety are represented by the complex projective plane* $\mathbf{C}P^2$*, the product of spheres* $S^2 \times S^2$ *and the connected sum of* $\mathbf{C}P^2$ *with a finite number of copies of* $-\mathbf{C}P^2$*.*

A Spectral sequences

A spectral sequence is a sequence of chain complexes each of which is the homology module of the preceding one. Associated to a spectral sequence there is a limit module, and the terms of the spectral sequence are regarded as approximations to this limit module. A spectral sequence can be used e.g. to approximate the cohomology of a topological space X by the relative cohomology modules of subspaces of X. Since this is the case in which we are interested, we describe cohomology spectral sequences rather than ordinary spectral sequences, though they are the same apart from the notation. Our exposition follows [6] and [9]. We first present the algebraic concept of spectral sequences.

In the following we consider modules over a fixed principal ideal domain. Let E be a bigraded module, i.e. a family of modules $(E^{s,t})_{s,t\in\mathbf{Z}}$. A differential $d : E \to E$ of bidegree $(r, 1-r)$ is a family of homomorphisms $d : E^{s,t} \to E^{s+r,t+1-r}$ such that $d^2 = 0$. The homology module $H(E, d)$ is the bigraded module defined by

$$H^{s,t}(E, d) = \ker[d : E^{s,t} \to E^{s+r,t+1-r}]/dE^{s-r,t-1+r}.$$

Definition A.1 A (cohomology) *spectral sequence* $(E_r, d_r)_{r\geq 1}$ is a sequence of bigraded modules E_r and differentials d_r of bidegree $(r, 1-r)$ such that

$$H(E_r, d_r) \cong E_{r+1} \quad \forall r.$$

The module E_1 is called the initial term of the spectral sequence. In order to define the limit term, we identify the module E_{r+1} with $H(E_r, d_r)$ by the isomorphism above. Then $E_2 = H(E_1, d_1)$ is a subquotient C_1/B_1 of E_1, where $C_1 = \ker d_1$ and $B_1 = \operatorname{im} d_1$. In turn $E_3 = H(E_2, d_2)$ is a subquotient of C_1/B_1 and hence it is isomorphic to C_2/B_2, where $B_2 \subset C_2$ are submodules of C_1 such that $C_2/B_1 = \ker d_2$ and $B_2/B_1 = \operatorname{im} d_2$. By induction we obtain a tower of submodules

$$B_1 \subset B_2 \subset B_3 \subset \ldots \subset C_3 \subset C_2 \subset C_1$$

of E_1 such that $E_{r+1} \cong C_r/B_r$. We set $B_\infty = \bigcup_r B_r$ and $C_\infty = \bigcap_r C_r$. Then $B_\infty \subset C_\infty$ and the quotient module $E_\infty = C_\infty/B_\infty$ is called the limit term of the spectral sequence.

The spectral sequence (E_r, d_r) is said to be convergent, if for every pair (s, t) there exists an index $r(s, t)$ such that the differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+1-r}$ are trivial for $r \geq r(s, t)$. In this case $E_{r+1}^{s,t}$ is isomorphic to a quotient of $E_r^{s,t}$, and $E_\infty^{s,t}$ is the direct limit of the sequence $E_{r(s,t)}^{s,t} \to E_{r(s,t)+1}^{s,t} \to \ldots$

A particular example of a convergent spectral sequence is a first quadrant spectral sequence, i.e. a spectral sequence for which $E_r^{s,t} = 0$ if $s < 0$ or $t < 0$ (for some r).

For such a spectral sequence there even exist indices $r(s,t)$ such that the homomorphisms in the sequence above are isomorphisms and hence $E_\infty^{s,t} \cong E_{r(s,t)}^{s,t}$.

We now study the spectral sequence associated to a filtration of a cochain complex. A (decreasing) filtration F of a module A is a sequence of submodules $(F^sA)_{s\in\mathbf{Z}}$ such that $F^sA \supset F^{s+1}A$. A filtration F determines an associated graded module $G(A)$ defined by $G(A)_s = F^sA/F^{s+1}A$. If $A = (A_t)_{t\in\mathbf{Z}}$ is a graded module, the submodules F^sA are graded by $(F^sA)_t = F^sA \cap A_t$ and the associated module $G(A)$ is bigraded by $G(A)_{s,t} = F^sA_{s+t}/F^{s+1}A_{s+t}$.

A (decreasing) filtration of a cochain complex $C^* = (C^q, \delta^q)$ is a sequence of cochain subcomplexes $(F^sC^*)_{s\in\mathbf{Z}}$ such that $F^sC^* \supset F^{s+1}C^*$. The filtration F is said to be bounded, if for every q there exist indices $s_1(q) < s_2(q)$ such that $F^{s_1(q)}C^q = C^q$ and $F^{s_2(q)}C^q = 0$.

Theorem A.1 *Let F be a filtration of a cochain complex $C^* = (C^q, \delta^q)$. There is a spectral sequence (E_r, d_r) with initial term*

$$E_1^{s,t} \cong H^{s+t}(F^sC^*/F^{s+1}C^*)$$

and differential d_1 corresponding to the coboundary operator of the triple $(F^sC^*, F^{s+1}C^*, F^{s+2}C^*)$. If F is bounded the spectral sequence is convergent, and the limit term E_∞ is isomorphic to the bigraded module associated to the filtration of the cohomology module $H(C^*)$ defined by

$$F^sH(C^*) = \mathrm{im}[H(F^{s-1}C^*) \to H(C^*)].$$

(In this case we say that the spectral sequence converges to the cohomology of C^*.)

Example A.1 Let X be a CW-complex and $(X_s)_{s\in\mathbf{Z}}$ an increasing filtration of X consisting of CW-subcomplexes. There is an induced (decreasing) filtration F of the cellular cochain complex $C^*(X)$ given by the subcomplexes

$$F^sC^*(X) = \{c \in C^*(X) |\ c|_{C_*(X_{s-1})} = 0\}$$

where $C_*(X)$ is the cellular chain complex of X. If for every q-skeleton X^q of X there exist indices $s_1(q) < s_2(q)$ such that $X_{s_1(q)} \subset X^q \subset X_{s_2(q)}$, the filtration F is bounded. In this case there is a convergent spectral sequence (E_r, d_r) with $E_1^{s,t} \cong H^{s+t}(X_s, X_{s-1})$, d_1 corresponding to the connecting homomorphism in the long exact cohomology sequence of the triple (X_{s+1}, X_s, X_{s-1}) and

$$E_\infty^{s,t} \cong \ker[H^{s+t}(X) \to H^{s+t}(X_{s-1})]/\ker[H^{s+t}(X) \to H^{s+t}(X_s)].$$

B Tensor products and exterior algebra

All modules in this appendix are modules over a fixed commutative ring R with a unit. Furthermore we assume that all R-algebras have a unit and that a homomorphism between two R-algebras is unit preserving.

A graded algebra consists of a graded module $A = \bigoplus_{q\in\mathbf{Z}} A_q$ and a homomorphism $\mu : A\otimes A \to A$ of degree 0, i.e. μ maps $A_s \otimes A_t$ to A_{s+t}. μ is called the product of the algebra and for $a, a' \in A$ we write $aa' = \mu(a \otimes a')$. The algebra A is commutative if $aa' = (-1)^{\deg a \deg a'} a'a$ for all $a, a' \in A$, where deg denotes the degree of an element $a \in A$.

If $A = \bigoplus_{q\in\mathbf{Z}} A_q$ and $B = \bigoplus_{q\in\mathbf{Z}} B_q$ are graded algebras, their tensor product $A \otimes B$ is graded too by $(A \otimes B)_q = \bigoplus_{s+t=q} A_s \otimes B_t$, and with the product

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg b \deg a'} aa' \otimes bb'$$

$A \otimes B$ becomes a graded algebra which is associative or commutative whenever A and B are.

Example B.1 Let R be a principal ideal domain. If X is a topological space, then $H^*(X; R) = \bigoplus_{q\geq 0} H^q(X; R)$ is a graded algebra whose product is the cup product. $H^*(X; R)$ is associative and commutative, it is called the cohomology algebra of X. If Y is a second topological space whose homology module $H_*(Y; R)$ is free and of finite type, there is an isomorphism of graded algebras

$$H^*(X; R) \otimes H^*(Y; R) \cong H^*(X \times Y; R)$$

by the Künneth theorem (see [9], 5.6.1).

Definition B.1 Let $\bigotimes^q A$ denote the q-fold tensor product of the module A with itself. (In particular we set $\bigotimes^0 A = R$ and $\bigotimes^1 A = A$.) Then a product is defined on the direct sum $\bigotimes^* A = \bigoplus_{q\geq 0} \bigotimes^q A$ by

$$ab = (\textstyle\sum_q a_q)(\sum_q b_q) = \sum_q(\sum_{s+t=q} a_s \otimes b_t)$$

where $a_q, b_q \in \bigotimes^q A$ are the homogeneous components of a and b, respectively. This makes $\bigotimes^* A$ an associative graded algebra, called the *tensor algebra* on A.

Definition B.2 Let $\bigotimes^* A$ be the tensor algebra on the module A and $I \subset \bigotimes^* A$ the ideal generated by the elements a^2, $a \in A$. Since I is graded, the quotient $\bigwedge^* A = \bigotimes^* A/I$ is graded too, and its component $\bigwedge^q A$ of degree q is isomorphic to $\bigotimes^q A$ modulo the submodule generated by the elements $a_1 \otimes \ldots \otimes a_q$ with $a_i = a_j$

for some $i \neq j$. (In particular $\bigwedge^0 A \cong R$ and $\bigwedge^1 A \cong A$.) The product on $\bigotimes^* A$ induces a product on $\bigwedge^* A$ which is denoted by $\wedge$. The relation $a \wedge a = 0$ implies that $a \wedge a' = -a' \wedge a$ for all $a, a' \in A$, and by induction on the degree it follows that the product $\wedge$ is commutative. Thus $\bigwedge^* A$ is an associative and commutative graded algebra, called the *exterior algebra* on A.

The exterior algebra $\bigwedge^* A$ can be characterised by the following **universal property**: If $f : A \to X$ is a homomorphism into an algebra X such that $(f(a))^2 = 0$ for all $a \in A$, then f can be extended uniquely to an algebra homomorphism from $\bigwedge^* A$ to X (where A has been identified with $\bigwedge^1 A \subset \bigwedge^* A$). This implies in particular that every homomorphism f between two modules A and B induces a unique homomorphism $f^\wedge : \bigwedge^* A \to \bigwedge^* B$ such that $f^\wedge|_A = f$.

The module structure of $\bigwedge^* A$ can be described as follows. If the module A is finitely generated by the elements $e_1, \ldots, e_n$, then $\bigwedge^* A$ is generated by the products

$$e_{i_1} \wedge \ldots \wedge e_{i_r} \qquad (1 \leq i_1 < \ldots < i_r \leq n)$$

Moreover, if A is free with basis $\{e_1, \ldots, e_n\}$, the products above form a basis of $\bigwedge^* A$ and hence $\operatorname{rk}(\bigwedge^* A) = 2^n$.

The following lemma we need to compute the cohomology algebra of a torus.

Lemma B.1 *If A, B are two modules, there is an isomorphism of graded algebras*

$$\bigwedge^* A \otimes \bigwedge^* B \cong \bigwedge^* (A \oplus B)$$

Example B.2 Let R be a principal ideal domain. By T^d we denote the d-dimensional torus $S^1 \times \ldots \times S^1$ (d factors). Since the homology module of the circle S^1 is free and of finite type, there is an isomorphism $H^*(T^d; R) \cong \bigotimes^d H^*(S^1; R)$ by the Künneth theorem. The cohomology algebra of the circle is generated by a single element $\alpha \in H^1(S^1; R)$ for which $\alpha \cup \alpha = 0$ and therefore $H^*(S^1; R) \cong \bigwedge^*(R)$. By the previous lemma it follows that $H^*(T^d; R)$ is isomorphic to the exterior algebra on the free module of rank d over R.

C Symmetric bilinear forms

The main purpose of this appendix is to present the classification theorem of indefinite integral symmetric bilinear forms. For a thorough treatment of symmetric bilinear forms see [7].

Let R be a commutative ring with a unit and X a finitely generated free R-module. We always assume a symmetric bilinear form $\beta : X \times X \to R$ to be non-singular, i.e. for each linear map $\varphi : X \to R$ there exists a uniquely determined element $y_0 \in X$ such that φ is equal to the map $x \mapsto \beta(x, y_0)$. (For such a symmetric bilinear form the notation $x \cdot y = \beta(x, y)$ is also used.) By the rank of a symmetric bilinear form we mean the rank of the space on which it is defined. Two symmetric bilinear forms $\beta_1 : X_1 \times X_1 \to R$ and $\beta_2 : X_2 \times X_2 \to R$ are isomorphic, if there exists an isomorphism $f : X_1 \to X_2$ such that

$$\beta_2(f(x), f(y)) = \beta_1(x, y) \qquad \forall\, x, y \in X_1.$$

In order to classify symmetric bilinear forms up to isomorphism, the following construction is useful.

Definition C.1 If $\beta_1 : X_1 \times X_1 \to R$ and $\beta_2 : X_2 \times X_2 \to R$ are two symmetric bilinear forms, their *orthogonal sum* is the symmetric bilinear form β which is defined on the direct sum $X_1 \oplus X_2$ by

$$\beta((x_1, x_2), (y_1, y_2)) = \beta_1(x_1, y_1) + \beta_2(x_2, y_2).$$

Example C.1 Let R be a principal ideal domain. If X is an oriented manifold of dimension $n = 2k$, then the intersection numbers of homology classes define a bilinear form on the free submodule of $H_k(X; R)$, called the intersection form of X. By Poincaré duality the intersection form is non-singular and if k is even, it is symmetric. If Y is another oriented manifold of even dimension, then the intersection form of the connected sum $X \# Y$ is the orthogonal sum of the intersection forms of X and Y.

We now restrict to integral symmetric bilinear forms, i.e. $R = \mathbf{Z}$. Before we can state the main theorem we need some more definitions.

Definition C.2 Let $\beta : X \times X \to \mathbf{Z}$ be a symmetric bilinear form.
(i) β is called *positive (negative) definite*, if $x \cdot x > 0$ ($x \cdot x < 0$) for all $x \in X$, and β is called *indefinite* if $x \cdot x$ can take on both positive and negative values.
(ii) Let $\{b_1, \ldots, b_n\}$ be an orthogonal basis of $\mathbf{Q} \otimes X$ with respect to the induced symmetric bilinear form. Then the *signature* of β is defined to be the difference

between the number of elements b_i with $b_i \cdot b_i > 0$ and the number of elements b_j with $b_j \cdot b_j < 0$.
(iii) β is said to be of *odd type* if there is an element $x \in X$ with $x \cdot x$ odd, and β is of *even type* if there is no such element.

It can be shown that for every indefinite symmetric bilinear form there exists an element $x \neq 0$ with $x \cdot x = 0$. This implies by induction that every indefinite symmetric bilinear form of odd type is isomorphic to the orthogonal sum of bilinear forms of rank 1. Finally, by reducing the case of indefinite symmetric bilinear forms of even type to the case of odd forms, one obtains the following

Theorem C.1 *Two indefinite integral symmetric bilinear forms are isomorphic if and only if they have the same rank, signature and type.*

Remark The classification of positive definite integral symmetric bilinear forms is much a harder problem. For further information see [7].

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Curriculum Vitae

I was born in Basel on June 6th, 1961 to Heidi and Joseph Fischli-Gass. When I was six years old we moved to Bern. In autumn 1980 I began my studies in mathematics at the University of Bern. As subsidiary subjects I enroled for computer science and astronomy. In spring 1988 I received my Master's degree. Afterwards I worked for half a year for Swiss Reinsurance in London. Since 1989 I have been employed as an assistant by the Mathematics Institute at the University of Bern. My doctoral work is part of a project supported by the National Science Foundation.